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Ferromagnetic Relaxation. III. Theory of Instabilities*

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In this paper a quantum-mechanical model is used to investigate various instability processes in ferromagnetic insulators. Time-dependent perturbation theory is employed to find the condition for which the rate of energy flowing into a magnon or phonon is equal to the rate at which it relaxes. This leads to a general instability criterion for various magnon-magnon, photon-magnon, and magnon-phonon processes.

I. INTRODUCTION

IN 1953, Damon[1] and Bloembergen and Wang[2] observed that the microwave resonant susceptibility of ferromagnets decreased at a power level which was lower than that predicted by the Bloch-Bloembergen equations. They also observed the presence of a subsidiary absorption peak at a dc field below that required for resonance. These effects were explained by Suhl[3] as arising from the unstable growth of spin waves. In particular, the subsidiary absorption peak is due to a process in which the unstable spin waves have a frequency $\omega_k=\omega/2$, where ω is the frequency of the applied microwave field. Similarly, the resonant susceptibility decline is produced by a process in which $\omega_k=\omega$ and is called the second-order process. Recently, another process has been found[4,5] in which unstable spin waves with $\omega_k=\omega/2$ are produced by means of a microwave field at a frequency ω which is applied parallel to the dc saturation field. This is called the first-order parallel pump instability.

These instabilities have all been studied classically by considering the coupled differential equations of motion for the uniform precession and the spin waves. In these previous analyses the nth order process is a parametric process arising through nonlinear terms in the spin wave equations of motion which depend upon the nth power of the amplitude of the driving mode. In the present paper we reformulate this problem in field variables by quantizing the Hamiltonian. This treatment is essentially a rate-equation formulation, the relation of which to the amplitude formulation has been discussed by Suhl and Fletcher.[6] In this formalism the nonlinear classical terms are interpreted as scattering processes.

This physical picture was first mentioned by Suhl.[7] A similar quantum-mechanical interpretation has also been applied to the Manley-Rowe relations by Weiss.[8] This approach is suggested here by the fact that it has proven so fruitful in dealing with the relaxation mechanisms of ferromagnetic insulators.[9] This technique leads easily to various additional instability processes. Although these additional processes may also be obtained from equations of motion, their solution is often more cumbersome, especially in the case of higher order processes. A somewhat similar approach was used by Loos[10] to obtain the second-order Suhl threshold. However, his treatment of magnon loss is not consistent with the classical approach and therefore his answer does not compare with Suhl's. Furthermore, Loos's approach is confined only to this magnon instability, whereas our development applies to any boson-boson process.

II. GENERAL FORMULATION

Before considering particular interactions, let us assume we have an interaction Hamiltonian $\mathcal{H}$ which involves products of boson operators. The eigenfunctions[11] $|n_{k_1},\cdots,n_{k_i},\cdots\rangle$ of such a Hamiltonian are represented by the number of bosons in each state. From the well-known commutation properties it can be shown that these operators have the following nonvanishing matrix elements:

$$\langle n_{k_1},\cdots n_{k_i}-1,\cdots|c_{k_i}|n_{k_1},\cdots n_{k_i},\cdots\rangle=(n_{k_i})^{1/2},\quad (1\text{a})$$

$$\langle n_{k_1},\cdots n_{k_i}+1,\cdots|c_{k_i}^\dagger|n_{k_1},\cdots n_{k_i},\cdots\rangle=(n_{k_i}+1)^{1/2}.\quad (1\text{b})$$

From time-dependent perturbation theory we find that the probability per unit time, TP, that a system initially in some state $|n_{k_1},n_{k_2},\cdots n_{k_i},\cdots\rangle\equiv|k\rangle$ makes a transition to another state $|n_{k'_1},n_{k'_2}\cdots n_{k'_i}\cdots\rangle\equiv|k'\rangle$ is

$$\mathrm{TP}=\frac{2\pi}{\hbar}\int|\langle k'|\mathcal{H}|k\rangle|^2\rho(E)\delta(E_{k'}-E_k)dE,\qquad(2)$$

where $\rho(E)$ is the density of states.

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[1] R. W. Damon, Rev. Mod. Phys. **25**, 239 (1953).
[2] N. Bloembergen and S. Wang, Phys. Rev. **93**, 72 (1954).
[3] H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).
[4] F. R. Morgenthaler, J. Appl. Phys. **31**, 95S (1960); and doctoral dissertation, Massachusetts Institute of Technology, 1960 (unpublished).
[5] E. Schlömann, J. J. Green, and U. Milano, J. Appl. Phys. **31**, 386S (1960).
[6] H. Suhl and R. C. Fletcher, J. Appl. Phys. **32**, 281 (1961).
[7] H. Suhl, in *Proceedings of International Conference on Solid State Physics in Electronics and Telecommunications, Brussels, 1958* (Academic Press Inc., New York, 1960), Vol. 3.
[8] M. T. Weiss, Proc. IRE **45**, 1012 (1957).
[9] See, for example, Parts I and II of this series: I. M. Sparks, R. Loudon, and C. Kittel, Phys. Rev. **122**, 791 (1961). II. P. Pincus, M. Sparks, and R. C. LeCraw, *ibid.* **124**, 1015 (1961).
[10] J. Loos, Czech J. Phys. **11**, 490 (1961).
[11] In this section k characterizes any boson state. In subsequent sections, however, we shall characterize magnons by k, photons by ν, and phonons by q.

In general, $\rho(E)$ is the product of the density of initial and final states. When the initial or final state consists of only one state, then

$$\int_{\substack{\text{initial}\\ \text{or final}}} \rho(E)dE=1.$$

In radiation theory this is known as the line-shape factor. In the problems dealt with in this paper both the initial and final states consist of one state. However, the initial state is either a high Q electromagnetic mode or a magnetic mode driven by such an electromagetic mode. Therefore, compared with the final state, this initial state is taken to be discrete.

Now consider the density of final states. In the equation of motion approach to these problems the relaxation of the mode amplitudes is included by adding an imaginary part to their frequency. It can be shown that this is mathematically equivalent to assuming that their line shapes are Lorentzian. Therefore, since we want to be able to compare some of our results with those obtained from equations of motion, we shall assume that our final-state density has the Lorentzian form

$$\rho(E)=\frac{1}{\pi\hbar}\frac{\eta_{k'}}{\eta_{k'}^2+(\omega_{k'}-\omega_k)^2}, \tag{3}$$

where $\eta_{k'}$ is the relaxation frequency of the final state. In the cases studied in this paper the final state consists of two bosons each having equal and opposite wave vectors. If we assume that their relaxation frequencies are independent of the direction of the wave vectors, then

$$\eta_{k'}=\eta_k+\eta_{-k}=2\eta_k. \tag{4}$$

In general, $\mathcal{H}$ produces scattering between various states, thereby changing their occupation numbers. In particular, we are interested in the part of $\mathcal{H}$ which produces scattering between the initially excited mode (microwave field, uniform precession, etc.) and the mode which goes unstable (magnon, phonon, etc.). If we denote the number of quanta in the initial mode by n_0, then the rate at which energy is scattered into the eventually unstable modes is

$$\hbar\omega_0\left(\frac{dn_0}{dt}\right)_{\text{scat}} =\Delta n_0\hbar\omega_0[\text{TP}_{n_0\to n_0+\Delta n_0}-\text{TP}_{n_0\to n_0-\Delta n_0}]. \tag{5}$$

The change in the number of quanta, Δn_0, depends upon the order of the scattering. Thus, for example, in any first-order process, $\Delta n_0=1$.

If the relaxation of a potentially unstable k mode can be described by a relaxation frequency η_k, then the rate at which energy leaves this mode is

$$\hbar\omega_k(2\eta_k)(n_k-\bar{n}_k), \tag{6}$$

where $\bar{n}_k$ is the thermal equilibrium occupation number and the factor of 2 reflects the quadratic relation between mode amplitude and energy. Instability occurs when the number of quanta in the k mode required to maintain equilibrium becomes infinite. This physical condition was first applied classically to the parallel pump instability by Kittel.[12]

Since we are interested in the lowest threshold, we take the maximum value of TP, which occurs for $\omega_{k'}=\omega_k$. This provides us with a general criterion for any boson-boson scattering. In the following sections we shall consider particular examples of this.

III. MAGNON-MAGNON INSTABILITIES

The Hamiltonian giving rise to magnon-magnon scattering consists of the magnetic Zeeman, dipolar, and exchange interactions. In subsequent sections we shall deal with certain interactions phenomenologically. For this purpose it is convenient to use the continuum description of the magnetization $\mathbf{M}(\mathbf{r})$. The magnetization is then an operator, related to a spin $\mathbf{S}_i$ at the point $\mathbf{r}=\mathbf{r}_i$ by $\mathbf{M}(\mathbf{r})=2\mu\sum_i \mathbf{S}_i\delta(\mathbf{r}-\mathbf{r}_i)$ and obeying the commutation rules

$$[M_x(\mathbf{r}),M_y(\mathbf{r}')]=i2\mu M_z(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'). \tag{7}$$

The effective moment μ is defined in terms of the spectroscopic splitting factor, g, and the Bohr magneton, μ_B, as $\mu=g\mu_B/2$. The Hamiltonian is

$$\mathcal{H}=-H_0\int M_z(\mathbf{r})d\mathbf{r}+\frac{1}{2}\int\int\left[\frac{\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} -3\frac{\mathbf{M}(\mathbf{r})\cdot(\mathbf{r}-\mathbf{r}')\mathbf{M}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5}\right]d\mathbf{r}'d\mathbf{r}+\frac{D}{2\gamma\hbar} \times\int[(\nabla M_x)^2+(\nabla M_y)^2+(\nabla M_z)^2]d\mathbf{r}. \tag{8}$$

Here, H_0 is the applied dc field, D is a phenomenological exchange parameter, and γ is the gyromagnetic ratio. This Hamiltonian is simplified by making a series of transformations. These were first applied to the discrete Hamiltonian by Holstein and Primakoff.[13] The resulting diagonalized Hamiltonian responsible for scattering of the uniform precession is

$$\mathcal{H}=\sum_k \hbar\omega_k c_k{}^\dagger c_k+\tfrac{1}{2}\sum_{k\neq0}\hbar(f_k c_0 c_k{}^\dagger c_{-k}{}^\dagger+\text{c.c.}) +\tfrac{1}{2}\sum_{k\neq0}\hbar(g_k c_0 c_0 c_k{}^\dagger c_{-k}{}^\dagger+\text{c.c.}) +\tfrac{1}{2}\sum_{k\neq0}\hbar(h_k c_0 c_0 c_0 c_k{}^\dagger c_{-k}{}^\dagger+\text{c.c.}), \tag{9}$$

[12] C. Kittel, "Lectures on Magnetism," University of Paris, College of Science, Orsay, France, 1960 (unpublished).
[13] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).

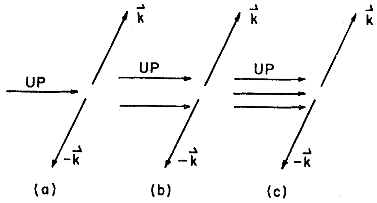

FIG. 1. Representations of the lowest order magnon-magnon instabilities (reference 15). (a) First order, $c_0c_k^\dagger c_{-k}^\dagger$; (b) second order, $c_0c_0c_k^\dagger c_{-k}^\dagger$; (c) third order, $c_0c_0c_0c_k^\dagger c_{-k}^\dagger$.

where c_k and $c_k^\dagger$ are the magnon annihilation and creation operators and

$$\omega_k=\left[\left(\gamma H+\frac{D}{\hbar}k^2\right)\left(\gamma H+\frac{D}{\hbar}k^2+\omega_M\sin^2\theta_k\right)\right]^{1/2}, \tag{10}$$

$$f_k=\left(\frac{4\mu}{M_sV}\right)^{1/2}\omega_M\frac{[\gamma H+(D/\hbar)k^2+\omega_k]}{2\omega_k}\times\sin\theta_k\cos\theta_k e^{-i\phi_k}, \tag{11}$$

$$g_k=\frac{4\mu}{M_sV}\frac{1}{8\omega_k}[\omega_k-\omega_0+(2-3\sin^2\theta_k)\omega_M]\times\left[\omega_k+\left(\omega_k^2+\frac{\sin^4\theta_k}{4}\omega_M^2\right)^{1/2}+\frac{1}{2}\omega_M^2\sin^4\theta_k\right], \tag{12}$$

$$h_k=\left(\frac{4\mu}{M_sV}\right)^{3/2}\frac{\omega_M}{16\omega_k}\left(\gamma H+\frac{D}{\hbar}k^2+\frac{1}{2}\omega_M\sin^2\theta_k+\omega_k\right)\times\sin\theta_k\cos\theta_k e^{-i\phi_k}. \tag{13}$$

In these expressions $H\equiv H_0-4\pi N_zM_s$, $\omega_M\equiv 4\pi\gamma M_s$, and θ_k and ϕ_k are the polar angles of the magnon k vector.

It should be noted that the canonical variables c_k and $c_k^\dagger$ are *not* the same as Suhl's classical amplitudes β_k but are related in the small amplitude approximation by

$$c_k=(M_sV/4\mu)^{1/2}\beta_k. \tag{14}$$

Using this fact with Hamilton's equations, the classical equations of motion could be obtained from (9). In fact, the general problem of ferromagnetic resonance at high power levels has also been analyzed in this fashion by Schlömann.[14]

We are now able to apply the general theory of Part II to various magnon processes.

A. First-Order Process

The first term f_k of the third-order term in the Hamiltonian (9) is represented schematically in Fig. 1(a).[15] Since this term depends linearly upon the $\mathbf{k}=0$ or uniform precession (UP) mode, it corresponds to a first-order process. By using this term as the interaction Hamiltonian in (2) and (5), we have, for the rate at which energy is scattered into the $\mathbf{k}$, $-\mathbf{k}$ magnon pair,

$$\hbar\omega_0\left(\frac{dn_0}{dt}\right)_{\text{scat}}=\frac{2\hbar\omega_0}{2\eta_k}|f_k|^2\times[(n_0+1)n_kn_{-k}-n_0(n_k+1)(n_{-k}+1)], \tag{15}$$

where $2\eta_k$ is appropriate to an *energy* relaxation and $2\hbar\omega_0$ is the uniform precession energy involved in the process. Just prior to the onset of instability, we have $1\ll(n_k,n_{-k})\ll n_0$. Therefore, the only term of importance in the bracket of (15) is $n_0(n_k+n_{-k})$.

The rate at which energy leaves the $\mathbf{k}$, $-\mathbf{k}$ pair by relaxation is

$$\hbar\omega_k\left(\frac{dn_k}{dt}\right)_{\text{relax}}=\hbar\omega_k(2\eta_k)[(n_k-\bar{n}_k)+(n_{-k}-\bar{n}_{-k})]. \tag{16}$$

Upon equating (15) to (16) and recalling that conservation of energy requires $\omega_k=\omega_0/2$, we find

$$n_k=[n_c/(n_c-n_0)]\bar{n}_k, \tag{17}$$

where

$$n_c=\eta_k^2/|f_k|^2. \tag{18}$$

Therefore, instability occurs when $n_0=n_c$.

The relation between the number of UP magnons excited and the amplitude of the circularly polarized transverse microwave field is found by equating the power absorbed to $n_0\hbar\omega(2\eta_0)$. The result is

$$h_0=\left(\frac{2\hbar[(\omega_0-\omega)^2+\eta_0^2]n_0}{\gamma M_sV}\right)^{1/2}. \tag{19}$$

Therefore, the magnon with the lowest threshold goes unstable when the microwave field reaches the value

$$h_{0_{\text{crit}}}=\min\left(\frac{2\omega_k\eta_k[(\omega_0-\omega)^2+\eta_0^2]^{1/2}}{\gamma\omega_M[\gamma H+(D/\hbar)k^2+\omega_k]\sin\theta_k\cos\theta_k}\right), \tag{20}$$

where min implies minimizing the expression in the brackets with respect to k and θ_k subject to the conservation of energy condition $\omega_k=\omega_0/2$. This tells us which pair of magnons goes unstable first. This expression agrees exactly with that obtained by Suhl from the spin wave equation of motion. This process is responsible for the subsidiary absorption peak mentioned in the Introduction.

B. Second-Order Process

Figure 1(b) shows the magnon scattering process that leads to the second order instability which arises from the fourth-order term of Eq. (9). In this process conservation of energy requires that $\omega_k=\omega_0$. This produces the premature saturation of the main resonance. By proceeding as in Sec. III A, we find that instability

[14] E. Schlömann, Technical Report No. R-48, Raytheon Company, 1959 (unpublished).

[15] In the figures of this paper, magnons are represented by straight arrows, photons by zig-zag arrows, and phonons by wavy arrows.

occurs when

$$h_{0_{\rm crit}}=\min\frac{2\hbar\eta_k[(\omega_0-\omega)^2+\eta_0^2]^{1/2}}{\gamma M_s V|g_k|}. \qquad (21)$$

For $\omega_M/\omega<1$ the expression for g_k, with $\omega_k=\omega$, may be approximated by

$$g_k\cong\frac{4\mu}{M_sV}\frac{\omega_M}{2}(1-\tfrac{3}{2}\sin^2\theta_k). \qquad (22)$$

Therefore, $h_{0_{\rm crit}}$ will be a minimum for a magnon having $\theta_k=0$, corresponding to a z-directed magnon. The critical field at resonance is then

$$h_{0_{\rm crit}}=\frac{\eta_0}{\gamma}\left(\frac{2\eta_k}{\omega_M}\right)^{1/2}. \qquad (23)$$

This threshold also agrees exactly with that obtained by Suhl and gives the value at which the resonant susceptibility begins its premature decline.

The agreement between our approach and that of Suhl for these processes is expected since both are essentially first-order analyses of the same Hamiltonian and have the same physical content.

C. Third-Order Process

The two processes discussed above have had direct experimental implications. We now consider the third-order process, the direct evidence for which has not as yet been observed. In this process energy conservation requires that the unstable magnons have a frequency $\omega_k=3\omega/2$. The possibility of this third-order instability has been discussed by Morgenthaler.[16]

The third-order process arises through the fifth-order term in the Hamiltonian, the first term of which is represented in Fig. 1(c). Proceeding as above,

$$h_{0_{\rm crit}}=\min\left[\left(\frac{16\omega_k\eta_k}{\omega_M[\gamma H+(D/\hbar)k^2+\frac{1}{2}\omega_M\sin^2\theta_k+\omega_k]\sin\theta_k\cos\theta_k}\right)^{1/3}\left(\frac{(\omega_0-\omega)^2+\eta_0^2}{\gamma^2}\right)^{1/2}\right]. \qquad (24)$$

In order to observe this process the experimental conditions should forbid the first- and second-order processes. This can be accomplished by operating at a microwave frequency which is below the magnon dispersion curve where there are no magnons degenerate with the pump frequency. However, the uniform precession resonant frequency cannot lie below this curve. This means that one must operate at a dc field higher than that required for resonance, thus driving the uniform precession off resonance. Although magnons could be excited by the first- and second-order processes under these conditions, the fact that these magnons are far off resonance greatly increases their thresholds. In order to make the $(\omega_0-\omega)$ factor in Eq. (24) as small as possible, the optimum experimental condition for observing this third-order instability should, therefore, employ a thin disk magnetized perpendicular to its plane.

IV. MAGNON-PHOTON INSTABILITIES: PARALLEL PUMPING

In the small signal region a microwave field applied parallel to the dc saturating field of a ferrite exhibits no absorption. However, at a certain threshold field the transverse susceptibility increases abruptly. This magnetic instability was suggested independently by Morgenthaler[4] and Schlömann *et al.*[5] The particular instability discussed by these authors corresponds to the case in which one photon excites two magnons, i.e., a first-order photon-magnon instability. In this section, we shall rederive this threshold, and also show that it is possible to have a second-order instability in which two photons excite two magnons. Evidence for such a second-order process has not as yet been observed experimentally.

We begin by quantizing the macroscopic microwave Zeeman interaction for the case in which the microwave field is applied parallel to the saturating dc field. This will entail quantizing the electromagnetic field to which the sample is exposed. Since the magnetic sample is usually placed in a cavity, the field quantization is carried out[17] in a slightly different manner than that for free space. In this case we expand the field in terms of the normal modes, $\mathbf{e}_\nu$, $\mathbf{h}_\nu$ of the cavity,

$$\mathbf{h}(\mathbf{r},t)=(4\pi)^{1/2}\textstyle\sum_\nu \omega_\nu q_\nu(t)\mathbf{h}_\nu(\mathbf{r}), \qquad (25)$$

$$\mathbf{e}(\mathbf{r},t)=(4\pi)^{1/2}\textstyle\sum_\nu p_\nu(t)\mathbf{e}_\nu(\mathbf{r}), \qquad (26)$$

which are orthogonal and normalized to the cavity volume V_c. The expansion parameters $q_\nu(t)$ and $p_\nu(t)$ are related to the photon field operators by

$$q_\nu=i[\hbar/2\omega_\nu]^{1/2}(c_\nu-c_\nu^\dagger), \qquad (27)$$

and

$$p_\nu=(2\hbar\omega_\nu)^{1/2}(c_\nu+c_\nu^\dagger). \qquad (28)$$

The c_ν's have the properties described by Eq. (1) with the n_ν's now referring to photons. These relations are defined such that the field energy, $E=(1/8\pi)\int(e^2+h^2)d\mathbf{r}$, takes the form (neglecting the zero-point energy),

$$E=\textstyle\sum_\nu \hbar\omega_\nu c_\nu^\dagger c_\nu. \qquad (29)$$

In writing the interaction between the field and the magnetic modes, let us consider, as an example, a small sample placed in a cavity which supports a mode ν. If the sample is small enough, only this mode will be

[16] F. R. Morgenthaler, J. Appl. Phys. **33**, 1297 (1962); and Air Force Cambridge Research Laboratories Technical Memorandum CRRD-54 (unpublished).

[17] See, for example, E. T. Jaynes, Microwave Laboratory Report No. 502, Stanford University, 1958 (unpublished).

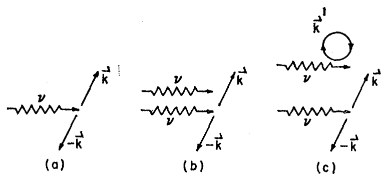

FIG. 2. Representations of the first- and second-order parallel pump instabilities (reference 15). (a) First order, $c_\nu c_k^\dagger c_{-k}^\dagger$; (b) second order, $c_\nu c_\nu c_k^\dagger c_{-k}^\dagger$; (c) second order, $c_\nu c_{k'} c_{k'}^\dagger c_\nu c_k^\dagger c_{-k}^\dagger$.

excited. Also, the field at the sample will be uniform and equal to h_0, which is some multiple of the normalization A. Therefore, we have

$$h_0 = i(2\pi\hbar\omega_\nu)^{1/2} A (c_\nu - c_\nu^\dagger). \qquad (30)$$

The macroscopic-microwave Hamiltonian is $\int h_0 M_z d\mathbf{r}$, which, after applying the first two Holstein-Primakoff transformations, becomes

$$\mathcal{H} = \int h_0 \left(M_s - \frac{2\mu}{V_s} \sum_{kk'} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} b_k^\dagger b_{k'} \right) d\mathbf{r}. \qquad (31)$$

Since h_0 is uniform, when the second term is integrated over the sample, it gives the Kronecker delta $\Delta(\mathbf{k}'-\mathbf{k})V_s$, where V_s is the volume of the sample. Therefore, the Hamiltonian is

$$\mathcal{H} = iAM_sV_s(2\pi\hbar\omega_\nu)^{1/2}(c_\nu - c_\nu^\dagger) - i2\mu A(2\pi\hbar\omega_\nu)^{1/2} \textstyle\sum_k (c_\nu - c_\nu^\dagger) b_k^\dagger b_k. \qquad (32)$$

Upon making the third Holstein-Primakoff transformation, this becomes

$$\begin{aligned}\mathcal{H} = {} & iAM_sV_s(2\pi\hbar\omega_\nu)^{1/2}(c_\nu - c_\nu^\dagger) \\ & - i2\mu A(2\pi\hbar\omega_\nu)^{1/2} \textstyle\sum_k (u_k^2 c_\nu c_k c_k^\dagger - u_k v_k^* c_\nu c_k c_{-k} \\ & - u_k v_k c_\nu c_k^\dagger c_{-k}^\dagger + |v_k|^2 c_\nu c_{-k} c_{-k}^\dagger + \text{c.c.}),\end{aligned} \qquad (33)$$

where

$$u_k = \left(\frac{\gamma H + (D/\hbar)k^2 + \frac{1}{2}\omega_M \sin^2\theta_k + \omega_k}{2\omega_k} \right)^{1/2} \qquad (34)$$

and

$$v_k = \left(\frac{\gamma H + (D/\hbar)k^2 + \frac{1}{2}\omega_M \sin^2\theta_k - \omega_k}{2\omega_k} \right)^{1/2}. \qquad (35)$$

A. First-Order Instability

The terms of Eq. (33) having the form $c_\nu c_k^\dagger c_{-k}^\dagger$ are responsible for a first-order instability. This process is shown in Fig. 2(a). Using the theory of Sec. II, the rate at which energy is scattered from the photon field into the $\mathbf{k}$, $-\mathbf{k}$ magnon pair is

$$\hbar\omega_\nu \left(\frac{dn_\nu}{dt} \right)_{\text{scat}} = \frac{8\hbar\omega_\nu [2\mu A(2\pi\hbar\omega_\nu)^{1/2}(u_k v_k)]^2}{\hbar^2 (2\eta_k)} \times [(n_\nu+1) n_k n_{-k} - n_\nu (n_k+1)(n_{-k}+1)]. \qquad (36)$$

By retaining only the $n_\nu(n_k+n_{-k})$ term for the same reasons used above and equating this to $\hbar\omega_k(2\eta_k) \times [(n_k - \bar{n}_k) + (n_{-k} - \bar{n}_k)]$, we find that magnon instability occurs when the number of photons reaches the value

$$n_\nu = \frac{\hbar^2 \eta_k^2}{4\mu^2 A^2 (8\pi\hbar\omega_\nu)(u_k v_k)^2}. \qquad (37)$$

Upon comparing the field energy $(1/8\pi)(h_0^2 A^2)$ with (29), we see that the amplitude of the magnetic field at the sample is related to the number of photons in the cavity by

$$h_0 = (8\pi\hbar\omega_\nu A^2 n_\nu)^{1/2}. \qquad (38)$$

Therefore, we have

$$h_{0\text{crit}} = \min \left[\frac{2\omega_\nu \eta_k}{\gamma\omega_M \sin^2\theta_k} \right]. \qquad (39)$$

Equation (39) agrees with the results of references 4 and 5.

B. Second-Order Instability

By analogy with the second-order Suhl instability we can see that a second-order parallel pump instability should involve two photons. However, the Hamiltonian (33) does not involve any such terms. Therefore, we must employ second-order time-dependent perturbation theory. The second-order matrix element has the form[18]

$$\sum_{k''} \frac{\langle k' | \mathcal{H} | k'' \rangle \langle k'' | \mathcal{H} | k \rangle}{E_k - E_{k''}}, \qquad (40)$$

where $|k\rangle$ is the initial state, $|k'\rangle$ the final state, and $|k''\rangle$ an intermediate state. In particular, the second-order process involves the destruction of two photons and the creation of two magnons with equal and opposite k vectors. Such a process involving the product of the linear terms of $\mathcal{H}$ with the cubic terms is shown in Fig. 2(b). Products among the cubic terms themselves can also produce such a process. This is illustrated in Fig. 2(c). The "loop" corresponds to the virtual creation and destruction of a magnon with any $\mathbf{k}$. It can be shown that the threshold for the process shown in Fig. 2(c) compared to that shown in Fig. 2(b) is larger by the factor $N/\sum_k \bar{n}_k$. At room temperature this is much larger than one. Therefore, we shall consider only the process of Fig. 2(b). As in the previous cases we will consider only those magnons which go unstable first. This eliminates the sum over $\mathbf{k}$ in the cubic part of the Hamiltonian as well as the sum over the intermediate states in (40). The linear part of the Hamiltonian must also be written in the "per mode" form by dividing it by the number of magnon mode pairs, $N/2$. Therefore, the

[18] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York), 2nd ed., p. 202.

matrix element for the second-order instability becomes

$$2\left\langle n_\nu-2,\, n_k+1,\, n_{-k}+1\left|\frac{i2AM_sV}{N}(2\pi\hbar\omega_\nu)^{1/2}c_\nu\right.\right.$$
$$\times\left|n_\nu-1,\, n_k+1,\, n_{-k}+1\right\rangle\langle n_\nu-1,\, n_k+1,\, n_{-k}+1|$$
$$\times i4\mu A\,(2\pi\hbar\omega_\nu)^{1/2}(u_kv_k)c_\nu c_k{}^\dagger c_{-k}{}^\dagger|n_\nu n_k n_{-k}\rangle$$
$$\times(\hbar\omega_\nu-2\hbar\omega_\nu)^{-1}. \quad (41)$$

By proceeding with the theory of Sec. II, we find

$$h_{0\mathrm{crit}}=\min\left[\frac{2\omega_\nu}{\gamma\sin\theta_k}\left(\frac{\eta_k}{\omega_M}\right)^{1/2}\right]. \quad (42)$$

This result agrees with unpublished equation of motion calculations by Morgenthaler[4] and Joseph *et al.*[19] However, the classical approach to these higher order processes involves certain mathematical difficulties as discussed in reference 19 which complicate their physical interpretation. In such cases a quantum-mechanical approach may be more illuminating.

V. MAGNON-PHONON INSTABILITIES

In this section we shall investigate the possibility of uniform precession magnons (excited by a transverse microwave field) producing phonons in a threshold process. This possibility was first suggested independently by Auld[20] and by Morgenthaler.[20]

For the purposes of this paper, we shall deal with the magnon-phonon coupling in a phenomenological fashion. In addition to the Hamiltonian (8) there are also anisotropy and elastic contributions:

$$\mathcal{H}_{\mathrm{anis}}=\sum_{NN'}K_{NN'}M_NM_{N'}+\cdots, \quad (43)$$

$$\mathcal{H}_{\mathrm{elastic}}=\sum_{ijkl}c_{ijkl}\epsilon_{jk}\epsilon_{kl}+\cdots. \quad (44)$$

The strain tensor ϵ_{ij} is defined as

$$\epsilon_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}, \quad (45)$$

where $\mathbf{u}(\mathbf{r})$ is the vector displacement operator in the continuum crystal.

In general, the anisotropy constants $K_{NN'}$ will be functions of the strain. Therefore, if we make a Taylor series expansion, the anisotropy Hamiltonian takes the form

$$\mathcal{H}_a=\sum_{NN'}K_{NN'}{}^{(0)}M_NM_{N'}+\sum_{ijNN'}b_{ijNN'}M_NM_{N'}\epsilon_{ij}$$
$$+\sum_{ijklNN'}g_{ijklNN'}{}^{(i)}M_NM_{N'}\epsilon_{ij}\epsilon_{kl}+\cdots. \quad (46)$$

The second term in this expansion is the familiar magnetoelastic interaction while the third term leads to the "intrinsic effect."[21]

Similarly, the elastic constants in (44) will be functions of the magnetization. By expanding these and using the fact that the terms must be invariant with respect to time reversal, we have[22]

$$\mathcal{H}_e=\sum_{ijkl}c_{ijkl}{}^{(0)}\epsilon_{ij}\epsilon_{kl}$$
$$+\sum_{ijklNN'}g_{ijklNN'}{}^{(m)}M_NM_{N'}\epsilon_{ij}\epsilon_{kl}+\cdots. \quad (47)$$

Notice that the second term in this expansion has the same form as the intrinsic term above. However, its origin is quite different and is referred to as the "morphic effect."

The transformation to elastic collective mode variables which reduce the elastic energy to the form $\sum_q\hbar\omega_qc_q{}^\dagger c_q$ is

$$\epsilon_{lm}=V^{-1/2}\sum_{\mathbf{q}s}\left[\left(\frac{\hbar}{2q\omega_{qs}}\right)^{1/2}(\hat{p}_{qs}\cdot\hat{x}_l)q_m(c_{qs}-c_{-qs}{}^\dagger)e^{i\mathbf{q}\cdot\mathbf{r}}\right.$$
$$\left.+\left(\frac{\hbar}{2\rho\omega_{qs}}\right)^{1/2}(\hat{p}_{qs}\cdot\hat{x}_m)q_l(c_{qs}-c_{-qs}{}^\dagger)e^{i\mathbf{q}\cdot\mathbf{r}}\right], \quad (48)$$

where $\mathbf{q}$ and ω_{qs} are the wave vector and frequency of the lattice vibration with polarization s, $\hat{p}_{qs}$ is the phonon polarization vector, $c_{qs}{}^\dagger$ and c_{qs} are the phonon creation and annihilation operators, and ρ is the density. The operators $c_{qs}{}^\dagger$ and c_{qs} act on phonon states in accordance with (1).

The magnon-phonon interaction is quantized by expanding the magnetization in magnon operators and the strain according to (48). We can see immediately that the strain is linear in phonon operators. Therefore, the magnetoelastic part of (46) describes one-phonon processes. Since we are interested in phonons which are produced from a uniform precession, momentum conservation would require $\mathbf{q}=0$ or $\hbar\omega_q=0$, thereby violating energy conservation. Consequently, this interaction does not lead to instabilities and we must either go to the next higher order terms in the expansion (the intrinsic and morphic terms) or apply second-order perturbation theory to the magnetoelastic part. Which process produces the lowest threshold depends upon the relative magnitudes of the coefficients in (46) and (47). An equation of motion calculation using the magnetoelastic and elastic interactions has been made by Auld *et al.*[22] A similar calculation using the combined morphic and intrinsic effects has been carried out by Morgenthaler.[23] To further illustrate the techniques developed in this paper, we shall investigate the instability thresh-

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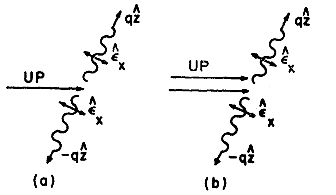

FIG. 3. Representation of the transverse phonon instabilities (reference 15). (a) First order, $c_0 c_q^\dagger c_{-q}^\dagger$; (b) second order, $c_0 c_0 c_q^\dagger c_{-q}^\dagger$.

olds for certain particular phonons excited through the morphic and intrinsic effects.

A. First-Order Transverse Phonon Instability

Consider first a transverse phonon linearly polarized in the x direction propagating in the z direction. For such a phonon, only $\epsilon_{xz}\neq 0$. Therefore, we have

$$\mathcal{H}=\int g_{ijxz} M_i M_j \epsilon_{xz}{}^2 d\mathbf{r}, \tag{49}$$

where g_{ijxz} is the appropriate constant containing both intrinsic and morphic contributions. The first-order process arises when i or j in (49) is z. In particular, let us consider $i=x$ and $j=z$. Expanding the magnetizations and strains gives

$$\mathcal{H}=\frac{\hbar M_s g}{2\rho}\left(\frac{\mu M_s}{V}\right)^{1/2}\sum_{kqq'}\frac{qq'}{(\omega_q\omega_{q'})^{1/2}}(b_k+b_{-k}{}^\dagger)\times(c_q-c_{-q}{}^\dagger)(c_{q'}-c_{-q'}{}^\dagger)\Delta(\mathbf{k}+\mathbf{q}+\mathbf{q}'). \tag{50}$$

In this expression the third Holstein-Primakoff transformation has not been applied to the magnon operators, b_k, since we shall only be interested in the uniform precession which we shall assume to be circularly polarized. Retaining only the terms in which a uniform precession magnon scatters with the phonons gives

$$\mathcal{H}=\frac{1}{2}\sum_q \hbar\frac{M_s gq}{\rho v_q}\left[\frac{\mu M_s}{V}\right]^{1/2}(b_0 c_q{}^\dagger c_{-q}{}^\dagger+\text{c.c.}), \tag{51}$$

where $v_q=\omega_q/q$ is the phonon velocity. This interaction is shown in Fig. 3(a). By using the methods of Sec. II, we find

$$\hbar\omega_0\left(\frac{dn_0}{dt}\right)_{\text{scat}}=\frac{\hbar\omega_0}{\eta_q}\left[\frac{M_s gq}{\rho v_q}\left(\frac{\mu M_s}{V}\right)\right]^{1/2}\times[(n_0+1)n_q n_{-q}-n_0(n_q+1)(n_{-q}+1)]. \tag{52}$$

The phonon relaxation is often conveniently described by an effective "Q" defined as

$$1/Q=\eta_q/\omega_q. \tag{53}$$

Therefore, we find that the number of UP magnons producing phonon instability is

$$n_0=\left[\frac{\rho v_q{}^2}{M_s gQ}\left(\frac{V}{\mu M_s}\right)\right]^{1/2}, \tag{54}$$

or, by (19),

$$h_{0_{\text{crit}}}=\frac{2}{\gamma}\left(\frac{\rho v_q{}^2}{M_s{}^2 gQ}\right)[(\omega_0-\omega)^2+\eta_0{}^2]^{1/2}. \tag{55}$$

B. Second-Order Transverse Phonon Instability

If neither i nor j are z in (49), we can have a second-order process. For example, consider $i=j=x$:

$$\mathcal{H}=g\int M_x{}^2\epsilon_{xz}{}^2 d\mathbf{r}. \tag{56}$$

After expanding and retaining only those terms involving two uniform precession magnons, we have

$$\mathcal{H}=\frac{1}{2}\sum_q \hbar\left[\frac{gq}{\rho v_q}\left(\frac{2\mu M_s}{V}\right)^2\right][b_0 b_0 c_q c_{-q}+\text{c.c.}]. \tag{57}$$

The first part of this interaction is represented in Fig. 3(b). By proceeding as above, we are led to

$$h_{0_{\text{crit}}}=\frac{2}{\gamma}\left[\frac{\rho v_q{}^2}{M_s{}^2 gQ}\right]^{1/2}[(\omega_0-\omega)^2+\eta_0{}^2]^{1/2}. \tag{58}$$

Similar results could also be obtained for longitudinal phonon instabilities.

VI. SUMMARY

We have shown that if a material involves "nonlinear" interactions among boson collective modes, it is possible to obtain instabilities. These processes are characterized by a growth of the number of bosons in the unstable mode, which in turn alters the nature of the energy absorption. In particular, we have discussed the various processes occurring in ferromagnetic insulators. Similar processes should be possible in other media. For example, other nonlinear quantum effects such as the Raman process in a two-level maser may be analyzed as inelastic photon scattering processes. Also, nonlinear processes in plasmas involving the interaction between plasmons and photons may lead to instabilities which could be predicted by the technique used in this paper.

We would like to emphasize that in this paper we have restricted ourselves to finding the threshold condition. Nothing has been said about the behavior above the threshold. In this region, the rate at which energy enters the unstable mode exceeds its relaxation rate; therefore, the number of bosons will grow. A quantum mechanical analysis of this situation may lead to a better understanding of the behavior of such systems above their threshold.

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